9/8/25 Lecture 7

A fn $T: \mathbb{R}^n \to \mathbb{R}^m$ is

- a <u>matrix transformation</u> if $\exists$ $m \times n$ matrix A such that
  $T(\vec{v}) = A\vec{v} \quad \forall \vec{v} \in \mathbb{R}^n$

- a <u>linear transformation</u> if
  - $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$
  - $T(c\vec{v}) = cT(\vec{v})$

  $\forall \vec{v}, \vec{w} \in \mathbb{R}^n, \ c \in \mathbb{R}$

Proposition - Every linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation.

Lemma (supporting fact) - Linear transformations "commute" with linear combinations. That is, if $T: \mathbb{R}^n \to \mathbb{R}^m$ is <u>linear</u> then
$$T(c_1\vec{v}_1 + \dots + c_K v_K) = c_1 T(\vec{v}_1) + \dots c_K T(\vec{v}_K)$$

Proof of Lemma:

$T(c_1\vec{v}_1 + c_2\vec{v}_2 + \dots c_K\vec{v}_K)$

$T(c_1\vec{v}_1) + T(c_2\vec{v}_2 + \dots c_K\vec{v}_K)$

$c_1 T(\vec{v}_1) + T(c_2\vec{v}_2) + T(c_3\vec{v}_3 + \dots c_K\vec{v}_K)$

$c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \dots c_K T(\vec{v}_K)$

Example

$T: \mathbb{R}^2 \to \mathbb{R}^3$ is linear

$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\1\end{pmatrix} \Rightarrow T\begin{pmatrix}-7\\0\end{pmatrix} = T\left(-7\begin{pmatrix}1\\0\end{pmatrix}\right) = -7\left(T\begin{pmatrix}1\\0\end{pmatrix}\right) = -7\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}-7\\7\\-7\end{pmatrix}$

What is $T\begin{pmatrix}0\\1\end{pmatrix}$? Problem because $\begin{pmatrix}0\\1\end{pmatrix}$ is not a multiple of $\begin{pmatrix}1\\0\end{pmatrix}$

What is $T\begin{pmatrix}0\\1\end{pmatrix}$?

* Using some info from above *

Suppose $T\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$$\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-1\\2\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = T\left[\frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-1\\2\end{pmatrix}\right]$$

$$= \frac{1}{2}T\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2}T\begin{pmatrix}-1\\2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{2}\\-\frac{1}{2}\\1\end{pmatrix}$$

Proof of Prop: Consider $\vec{v} = \begin{pmatrix}v_1\\\vdots\\v_n\end{pmatrix} \in \mathbb{R}^n$, then

$$T(\vec{v}) = T\left[v_1\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} + v_2\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix} + v_n T\begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}\right]$$

$$= v_1 T\underbrace{\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}}_{=\vec{e}_1} + v_2 T\underbrace{\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}}_{=\vec{e}_2} + \dots + v_n T\underbrace{\begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}}_{=\vec{e}_n}$$

$$= \begin{bmatrix}T(\vec{e}_1) & \dots & T(\vec{e}_n)\end{bmatrix}\begin{bmatrix}v_1\\\vdots\\v_n\end{bmatrix}$$

So T is a matrix transformation and its matrix is

$$A = \begin{bmatrix}T(\vec{e}_1) & \dots & T(\vec{e}_n)\end{bmatrix} = [T]$$

$\boxed{\text{e.g.}}$ Fix an angle $\theta \in \mathbb{R}$ and let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the fn such that $R_\theta(\vec{v}) = \vec{v}$ rotated counterclockwise by θ. Then R_θ is a linear transformation and its matrix is

$$[R_\theta] = \left[R_\theta\begin{pmatrix}1\\0\end{pmatrix} \quad R_\theta\begin{pmatrix}0\\1\end{pmatrix} \right]$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$\boxed{\text{e.g}}$ $[T] = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix} \Rightarrow$ T is rotation by $\underline{\theta = \cos^{-1}\left(\frac{3}{5}\right)}$

☆ Linear transformation and matrix transformations are the same thing.

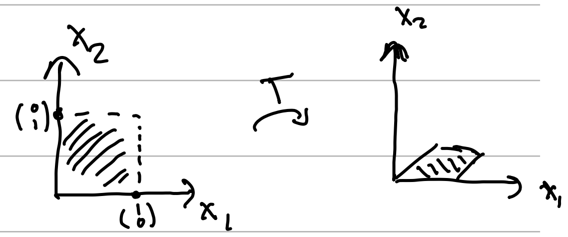
Other geometric transformation that are linear

- $\underline{\text{Scaling}}$ $T_\lambda : \mathbb{R}^3 \to \mathbb{R}^3$ def'd by $T_\lambda(\vec{v}) = \lambda \vec{v}$

  has matrix $[T_\lambda] = [T_\lambda(\vec{e}_1) \; \dots \; T_\lambda(\vec{e}_n)]$

  $= [\lambda \vec{e}_1 \; \dots \; \lambda \vec{e}_n]$

$\boxed{\text{Eg}}$ Fix $\lambda \in \mathbb{R}$, consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $[T] = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}$



"horizontal shear"